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Probabilistic Topological Spaces*

M. J. FRANK

Illinois Institute of Technology, Chicago, Illinois 60616

Submitted by L. A. Zadeh

1. INTRODUCTION

Often physical constructs are described by mathematical systems which are over-idealizations in a fundamental sense: quantities or concepts which are inherently imprecise in the "real" situation are translated into precise terms in the abstract system. For example, an abstract metric space frequently serves as a mathematical framework for physical systems even though a "distance" between two points usually results from an averaging process or an approximation rather than from the assignment of a single number.

In recent years a number of axiomatic systems have been generalized to embrace uncertainties in various physical schemes. Two of these will be of special interest here. Menger [2] generalized the metric axioms by associating a distribution function with each pair of points of a set. This system, called a *probabilistic metric space* (originally, a *statistical metric space*), has been developed extensively by Schweizer and Sklar [5], and later by others. Zadeh [10] introduced the *fuzzy set*, through which the usual notion of a set is generalized by the association of an element of the unit interval of real numbers with each member of a collection of objects. This concept corresponds to the physical situation in which there is no exact criterion for membership.

The purpose of this article is to generalize the topological structure on a set in the same spirit. To this end it seems reasonable to demand that such a system meet the following requirements: (1) the axioms should reflect those of a topological space; (2) just as a collection of subsets characterizes a topology, an appropriate collection of fuzzy sets should characterize our structure; (3) just as a metric determines a topology, a probabilistic metric should determine such a structure. Wagner [9] introduced a system of generalized topological axioms which fits these requirements, but which is too restrictive to include a natural class of spaces generated by collections of topologies. Appropriate modifications of those axioms will yield the desired generalization, which we shall call a *probabilistic topological space.*

We begin by formulating an axiomatization for a probabilistic topological

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space in terms of a generalized closure operator. Next we demonstrate fundamental properties of the system, including its characterization by fuzzy sets. After exhibiting several examples, we present an operator with which every probabilistic metric space satisfies our axioms. We then explore in some detail a class of examples which we shall call *topologically generated.* Finally, we state an equivalent formulation for the system in terms of a generalized interior operator.

We adopt some of Wagner's notation and modify several of his results in Sections 2, 4, and 6. Throughout, we refer to previously stated theorems by section and number (e.g., Theorem 2 of Section 5 is denoted 5.2).

For the sake of brevity, it is convenient to reserve certain symbols for objects which occur frequently. We use the Greek letters λ, μ, and ν for elements of I—the closed unit interval of real numbers—and α and β for real numbers. We denote an abstract set by the letter X, elements of X by the letters x, y, and z, the power set of X by $\mathscr{P}(X)$, and elements of $\mathscr{P}(X)$ by the letters A and B. We employ Γ for an indexing set and γ for an element of Γ. The symbol $\subseteq$ means "is contained in", and $\subset$ means "is strictly contained in". We refer to the topological closure of A (in the sense of Kuratowski) as Cl(A).

2. The θ-Closure Operator

A function $T : I \times I \to I$ is called a *t-function* if

(T1) $T(\lambda_1, \mu_1) \leqslant T(\lambda_2, \mu_2)$ whenever $\lambda_1 \leqslant \lambda_2$, $\mu_1 \leqslant \mu_2$.

A t-function T_1 is *weaker* than a t-function T_2 (equivalently, T_2 is *stronger* than T_1) if $T_1(\lambda, \mu) \leqslant T_2(\lambda, \mu)$ for every pair (λ, μ) and $T_1(\lambda, \mu) < T_2(\lambda, \mu)$ for at least one pair (λ, μ). We write $T_1 < T_2$.

The following t-functions will be of particular importance:

$$T_z(\lambda, \mu) = 0;$$

$$T_u(\lambda, \mu) = \begin{cases} \lambda, & \mu = 1 \\ 0, & \mu < 1; \end{cases}$$

$$T_w(\lambda, \mu) = \begin{cases} \lambda, & \mu = 1 \\ \mu, & \lambda = 1 \\ 0 & \text{otherwise}; \end{cases}$$

$$T_m(\lambda, \mu) = \max[\lambda + \mu - 1, 0];$$

$$\operatorname{Min}(\lambda, \mu) = \begin{cases} \lambda, & \lambda \leqslant \mu \\ \mu, & \lambda > \mu; \end{cases}$$

$$\operatorname{Max}(\lambda, \mu) = \begin{cases} \mu, & \lambda \leqslant \mu \\ \lambda, & \lambda > \mu; \end{cases}$$

$$T_s(\lambda, \mu) = 1.$$

(Note that $T_z < T_u < T_w < T_m < \operatorname{Min} < \operatorname{Max} < T_s$.)

A θ-*closure* on a set X is a mapping $\theta : \mathscr{P}(X) \times I \to \mathscr{P}(X)$ (we write $\theta(A, \lambda) = A^\lambda$) satisfying

(C1) $A^0 = X$ for every A;

(C2) $\varnothing^\lambda = \varnothing$ for every $\lambda > 0$;

(C3) $A \subseteq A^\lambda$ for every A, λ;

(C4) $A^\lambda \subseteq B^\lambda$ for every λ whenever $A \subseteq B$;

(C5) $A^\lambda \supseteq A^\mu$ for every A whenever $\lambda \leqslant \mu$.

If θ is a θ-closure on X and T is a t-function, then the ordered triple (X, θ, T) is a *probabilistic topological space* (PT space) if

(C6) $(A^\mu)^\lambda \subseteq A^{T(\lambda,\mu)}$ for every $A \neq \varnothing$ and every pair (λ, μ).

A θ-closure is *left-continuous at* λ if

(LC) $A^\lambda = \bigcap_{\mu<\lambda} A^\mu$ for every A,

and is *left-continuous* if it is left-continuous at λ for every $\lambda \neq 0$.

Elementary consequences

1. Axioms (C1)-(C6) are generalizations of the Kuratowski closure axioms under the interpretation "x is in A^λ iff the probability that x is in $\mathrm{Cl}(A)$ is greater than or equal to λ." However, it is important to note that since the axioms are free of any probabilistic notions, this interpretation is not essential to the validity of our results.

2. It is easy to verify that (C4) is equivalent to either of the following:

(C4′) $A^\lambda \cup B^\lambda \subseteq (A \cup B)^\lambda$ for every A, B, λ;

(C4″) $(A \cap B)^\lambda \subseteq A^\lambda \cap B^\lambda$ for every A, B, λ.

3. Since $(A^\mu)^\lambda \subseteq X = A^0$ for every A and every pair (λ, μ), (C6) is always satisfied with $T = T_z$. Hence if θ is a θ-closure on X, then (X, θ, T_z) is a PT space.

4. If (X, θ, T_2) is a PT space and $T_1 < T_2$, then (X, θ, T_1) is a PT space since by (C5) $(A^\mu)^\lambda \subseteq A^{T_2(\lambda,\mu)} \subseteq A^{T_1(\lambda,\mu)}$ for every A and every pair (λ, μ).

5. For any fixed $\lambda \neq 0$ the map $A \to A^\lambda$ is a Fréchet closure. But in order that $(A^\lambda)^\lambda = A^\lambda$ for every A, (C6) must be satisfied with a T for which $T(\lambda, \lambda) \geqslant \lambda$. Now $T(\lambda, \lambda) \geqslant \lambda$ for every λ iff $T \geqslant \mathrm{Min}$, a very strong condition.

6. If (X, θ, T) is a PT space and $T(0, \lambda) \geqslant \lambda$ for every λ, then $A^\lambda = X$ for every $A \neq \varnothing$, λ. For if there exists $A \neq \varnothing$ and λ such that $A^\lambda \subset X$, then $(A^\lambda)^0 = X \supset A^\lambda \supseteq A^{T(0,\lambda)}$, whence (C6) is not satisfied with T. Consequently, a θ-closure is "indiscrete" if (C6) is satisfied with any t-function for which $T(0, \lambda) \geqslant \lambda$—for example, $T = \mathrm{Max}$.

7. One inclusion of (LC) is universal since by (C5) $A^\lambda \subseteq \bigcap_{\mu<\lambda} A^\mu$ for every $\lambda \neq 0$. Although the existence of non-left-continuous θ-closures will be demonstrated, our results will indicate that "nice" θ-closures are left-continuous.

Given a θ-closure on X, we define the mapping $\underset{\sim}{c} : X \times \mathscr{P}(X) \to I$ by

$$c(x, A) = \sup\{\lambda : x \in A^\lambda\} \qquad \text{for each } x, A.$$

Consistent with the probabilistic interpretation of A^λ, we interpret "$c(x, A)$ is the probability that x is in $\mathrm{Cl}(A)$."

PROPOSITION 1. *A θ-closure on X is left-continuous iff*

$$A^\lambda = \{x : c(x, A) \geqslant \lambda\}$$

for every A, λ.

Proof. It is sufficient to show that $\bigcap_{\mu<\lambda} A^\mu = \{x : c(x, A) \geqslant \lambda\}$ for every A, $\lambda \neq 0$. If $c(x, A) \geqslant \lambda$, then $x \in A^\mu$ for every $\mu < \lambda$. If $x \in A^\mu$ for every $\mu < \lambda$, then $c(x, A) \geqslant \mu$ for every $\mu < \lambda$, whence $c(x, A) \geqslant \lambda$.

THEOREM 2. *If θ is a θ-closure on X, then $\underset{\sim}{c}$ satisfies*

(L1) $c(x, \varnothing) = 0$ *for every* x;

(L2) $c(x, A) = 1$ *for every* $x \in A$;

(L3) $c(x, A) \leqslant c(x, B)$ *for every x whenever* $A \subseteq B$.

If (X, θ, T) is a PT space, then

(L4) $c(x, A^\mu) > \lambda$ *implies* $c(x, A) \geqslant T(\lambda, \mu)$ *for every* $A \neq \varnothing$, μ.

If, in addition, θ is left-continuous, then

(L5) $c(x, A^\mu) \geqslant \lambda$ *implies* $c(x, A) \geqslant T(\lambda, \mu)$ *for every* $A \neq \varnothing$, μ.

THEOREM 3. *Let X be a set and $c : X \times \mathscr{P}(X) \to I$ be a mapping satisfying* (L1), (L2), *and* (L3). *Define a mapping θ by $\theta(A, \lambda) = \{x : c(x, A) \geqslant \lambda\}$ for each A, λ. Then θ is a left-continuous θ-closure on X. If, in addition, $\underset{\sim}{c}$ satisfies* (L5) *with t-function T, then (X, θ, T) is a PT space. Further by* 2.1 *$c(x, A) = \sup\{\lambda : x \in A^\lambda\}$ for every x, A.*

These results follow from the definition of $\underset{\sim}{c}$, (C1)-(C6), and (LC) by direct computation.

Frequently it is convenient to construct a θ-closure by defining $\underset{\sim}{c}$ and verifying (L1)-(L3). In such cases it will be assumed tacitly that the θ-closure intended is the one obtained from $\underset{\sim}{c}$ in 2.3.

If Y is a collection of objects, then a *fuzzy set* is a function $f_S : Y \to I$. For y in Y, $f_S(y)$ is called the *grade of membership* of y in f_S. The concept of a fuzzy set generalizes that of an ordinary set in the sense that the function f_S may be considered a generalized characteristic function. (For details, see Zadeh [10].)

We now show that the mapping $\underline{c}$ yields a characterization of left-continuous PT spaces in terms of certain collections of fuzzy sets. Given a θ-closure on X, for each $A \subseteq X$ let f_A be the fuzzy set in X defined by $f_A(x) = c(x, A)$ for each x. We call f_A the *closure cloud about A induced by θ*. Conversely, as a direct consequence of 2.3, we have

COROLLARY 3.1. *Given a set X, suppose there is a mapping f associating with each $A \subseteq X$ a fuzzy set f_A satisfying*

(i) $f_{\varnothing}(x) = 0$ *for every* x;

(ii) $f_A(x) = 1$ *for every* x *in* A;

(iii) $f_A(x) \leqslant f_B(x)$ *for every* x *whenever* $A \subseteq B$.

Then the mapping $\theta(A, \lambda) = \{x : f_A(x) \geqslant \lambda\}$ for each A, λ is a left-continuous θ-closure on X.

3. EXAMPLES

PROPOSITION 1. *Let $(X, \mathscr{T})$ be a topological space, and let*

$$\theta(A, \lambda) = \begin{cases} X, & \lambda = 0 \\ \mathrm{Cl}(A), & \lambda \neq 0 \end{cases} \quad \text{for each } A.$$

Then (X, θ, T) is a left-continuous PT space with

$$T(\lambda, \mu) = \begin{cases} 1, & \lambda, \mu \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Further, (X, θ, T_3) is a PT space iff $\mathscr{T}$ is indiscrete.

Proof. Clearly (C1)-(C5) and (LC) are satisfied. If $\lambda, \mu \neq 0$, then $(A^\mu)^\lambda = (A^\lambda)^\mu = \mathrm{Cl}(A) = A^1$, whence (C6) is satisfied with $T(\lambda, \mu) = 1$. If $\mathscr{T}$ is indiscrete, then $A^\lambda = X$ for every $A \neq \varnothing$, λ. Thus

$$(A^0)^\lambda = (A^\lambda)^0 = X = A^1,$$

whence (C6) is satisfied with $T(\lambda, 0) = T(0, \lambda) = 1$ for every λ. If $\mathscr{T}$ is not indiscrete, then there exists an $A \neq \varnothing$ such that $\mathrm{Cl}(A) \subset X$. Thus $(A^0)^\lambda = X \supset A^\nu$ for every $\nu \neq 0$, whence (C6) is not satisfied with $T(\lambda, 0) > 0$.

PROPOSITION 2. *If X is any set with more than two elements, then there exists a θ-closure on X for which (X, θ, T) is a PT space only if $T = T_z$.*

Proof. Let x and y be distinct fixed elements of X. Define θ by

$$\theta(A, \lambda) = \begin{cases} \varnothing, & A = \varnothing, \quad \lambda \neq 0 \\ \{x\} \cup \{y\}, & A = \{x\}, \quad \lambda \neq 0 \\ X, & \text{otherwise.} \end{cases}$$

Clearly θ is a θ-closure. But $(\{x\}^1)^1 = (\{x\} \cup \{y\})^1 = X \supset \{x\} \cup \{y\} = \{x\}^\nu$ for every $\nu \neq 0$. Thus (C6) is not satisfied with any t-function for which $T(1, 1) > 0$. Thus (X, θ, T) is a PT space only if $T(1, 1) = 0$, whence by (T1) $T = T_z$.

Since any set can be topologized with the indiscrete topology, these examples show that θ-closures can be defined on any set of more than two elements in such a way that (C6) is satisfied first with the strongest t-function and second, only with the weakest t-function.

We conclude this section by exhibiting a non-left-continuous θ-closure.

PROPOSITION 3. *Let $(X, \mathscr{T})$ be a non-indiscrete topological space. For $\lambda_0 \neq 0$ let*

$$\theta(A, \lambda) = \begin{cases} \mathrm{Cl}(A), & \lambda \geqslant \lambda_0 \\ X, & \lambda < \lambda_0 \end{cases}$$

for every $A \neq \varnothing$, and

$$\theta(\varnothing, \lambda) = \begin{cases} \varnothing, & \lambda \neq 0 \\ X, & \lambda = 0. \end{cases}$$

Then θ is a θ-closure on X which is not left-continuous at λ_0.

Proof. Clearly (C1)-(C5) are satisfied. Since $\mathscr{T}$ is non-indiscrete, there exists $A \neq \varnothing$ such that $\mathrm{Cl}(A) \subset X$. For every $\mu < \lambda_0$,

$$A^{\lambda_0} = \mathrm{Cl}(A) \subset X = A^\mu.$$

Thus $A^{\lambda_0} \subset \bigcap_{\mu < \lambda_0} A^\mu$, whence (LC) is not satisfied with $\lambda = \lambda_0$.

4. PROBABILISTIC METRIC SPACES

By a ***distribution function*** we mean a function from the real numbers into I which is non-decreasing and continuous from the left with inf 0 and sup 1. A *probabilistic metric space* (PM space) is an ordered pair $(X, \mathscr{F})$, where X is a set, and $\mathscr{F}$ is a mapping from $X \times X$ into the set of distribution functions (we write $\mathscr{F}(x, y) = F_{xy}$) satisfying

(PM1) $F_{xy}(\alpha) = 1$ for every $\alpha > 0$ iff $x = y$;

(PM2) $F_{xy}(0) = 0$ for every x, y;

(PM3) $F_{xy} = F_{yx}$ for every x, y;

(PM4) $F_{xz}(\alpha + \beta) = 1$ whenever $F_{xy}(\alpha) = F_{yz}(\beta) = 1$.

A PM space is a generalized metric space under the interpretation "$F_{xy}(\alpha)$ is the probability that the distance between x and y is less than α;" however, as with a PT space, the axioms are not probabilistic in nature.

A t-function T is a *t-norm* if it satisfies

(T2) $T(0, 0) = 0$, $T(\lambda, 1) = \lambda$ for every λ;

(T3) $T(\lambda, \mu) = T(\mu, \lambda)$ for every λ, μ;

(T4) $T(\lambda, T(\mu, \nu)) = T(T(\lambda, \mu), \nu)$ for every λ, μ, ν.

(Note that T_w, T_m, and Min are t-norms.)

An ordered triple $(X, \mathscr{F}, T)$ is a *Menger space* if $(X, \mathscr{F})$ is a PM space, T is a t-norm, and for every x, y, z and every $\alpha, \beta \geqslant 0$

(PM4M) $F_{xz}(\alpha + \beta) \geqslant T(F_{xy}(\alpha), F_{yz}(\beta))$.

The Menger inequality (PM4M) is a triangle inequality for a PM space. While Schweizer and Sklar [5] have shown that (T2)-(T4) are reasonable restrictions for a t-function in (PM4M), 3.1, 3.2, and certain later results illustrate the fact that these conditions are unnecessarily confining for (C6). Note that if $(X, \mathscr{F}, T_2)$ is a Menger space and $T_1 < T_2$, then $(X, \mathscr{F}, T_1)$ is a Menger space.

We now show that our axiom system for a PT space is a generalization of that for a Menger space.

THEOREM 1. *Let $(X, \mathscr{F})$ be a* PM *space, and define a mapping $\underline{c}'$ by*

$$c'(x, A) = \begin{cases} \inf\limits_{\alpha>0} \sup\limits_{y\in A} F_{xy}(\alpha), & A \neq \varnothing \\ 0, & A = \varnothing \end{cases}$$

for each x. Then (X, θ', T_z) is a left-continuous PT *space. If, in addition, $(X, \mathscr{F}, T)$ is a Menger space, then (X, θ', T) is a* PT *space.*

Proof.

(L1) $c'(x, \varnothing) = 0$ by hypothesis.

(L2) If $x \in A$, then

$$c'(x, A) = \inf_{\alpha>0} \sup_{y \in A} F_{xy}(\alpha) = \inf_{\alpha>0} (1) = 1$$

by (PM1).

(L3) If $A \subseteq B$, then

$$\begin{aligned} c'(x, A) &= \inf_{\alpha>0} \sup_{y\in A} F_{xy}(\alpha) \\ &\leqslant \inf_{\alpha>0} \sup_{y\in B} F_{xy}(\alpha) \\ &= c'(x, B). \end{aligned}$$

(C6) For $A \neq \varnothing$, $A^\lambda = \{x : \text{for every } \alpha > 0 \text{ there exists } y \in A \text{ such that } F_{xy}(\alpha) \geqslant \lambda\}$. Suppose $x \in (A^\mu)^\lambda$. Then for every α_1, $\alpha_2 > 0$ there exists $y \in A^\mu$ such that $F_{xy}(\alpha_1) \geqslant \lambda$ and then $z \in A$ such that $F_{yz}(\alpha_2) \geqslant \mu$. Given $\alpha > 0$, let $\alpha_1 = \alpha_2 = \alpha/2$. Then for some $z \in A$,

$$F_{xz}(\alpha) \geqslant T\left(F_{xy}\left(\frac{\alpha}{2}\right), F_{yz}\left(\frac{\alpha}{2}\right)\right) \geqslant T(\lambda, \mu)$$

by (PM4M) and (T1). Thus $x \in A^{T(\lambda,\mu)}$.

It can be shown for θ' that $A^\lambda \cup B^\lambda = (A \cup B)^\lambda$ for every A, B, λ. On the other hand, we shall investigate next a class of spaces which conforms to our intuitive notion of a generalized topology but which in general does not satisfy this stronger condition.

5. Topologically Generated Spaces

We now explore a class of PT spaces for which A^λ has the probabilistic interpretation mentioned in Section 2. We begin by summarizing briefly some results from a class of PM spaces developed from the same point of view.

An ordered triple $(\Omega, \mathcal{O}\!\mathcal{l}, P)$ is a *probability space* if Ω is a set, $\mathcal{O}\!\mathcal{l}$ is a σ-algebra of subsets of Ω, and $P : \mathcal{O}\!\mathcal{l} \to I$ is a function such that $P(\Omega) = 1$ and such that if $\{A_i\}_{i=1}^\infty$ is any collection of disjoint elements of $\mathcal{O}\!\mathcal{l}$, then $P(\cup A_i) = \sum P(A_i)$. P is called a *probability measure* on Ω; and if $A \in \mathcal{O}\!\mathcal{l}$, then A is said to be P-measurable.

Kolmogorov [1] has shown that if $(\Omega, \mathcal{O}\!\mathcal{l}, P)$ is a probability space, ψ a function from Ω into Ω', $\mathcal{O}\!\mathcal{l}' = \{A' \subseteq \Omega' : \psi^{-1}(A') \in \mathcal{O}\!\mathcal{l}\}$, then the function $P' : \mathcal{O}\!\mathcal{l}' \to I$ defined by $P'(A') = P(\psi^{-1}(A'))$ is a probability measure, which we call the *probability measure on Ω' induced from P by ψ.*

Stevens [7] has shown that if $\mathscr{D} = \{d_\gamma\}_{\gamma \in \Gamma}$ is a collection of metrics on a set X, P is a probability measure on a σ-algebra of subsets of $\mathscr{D}$ such that (i) $P(\mathscr{D}) = 1$ and (ii) $P\{d_\gamma : d_\gamma(x, y) < \alpha\}$ is P-measurable for every x, y, α, and if for each x, y, α, we define $F_{xy}(\alpha) = P\{d_\gamma : d_\gamma(x, y) < \alpha\}$, then $(X, \mathscr{F}, T_m)$ is a Menger space, called a *metrically generated space*. Further, if $\mathscr{D}$ is linearly ordered by "$d_i < d_j$ iff $d_i(x, y) \leqslant d_j(x, y)$ for every pair (x, y)", then $(X, \mathscr{F}, \mathrm{Min})$ is a Menger space.

Sherwood [6] has shown that if $(\Omega, \mathscr{A}, P)$ is a probability space, X is a collection of functions from Ω into a metric space (M, d) such that $\{t \in \Omega : d(x(t), y(t)) < \alpha\}$ is P-measurable for every x, y, α, and for each x, y, α we define $F_{xy}(\alpha) = P\{t : d(x(t), y(t)) < \alpha\}$, then $(X, \mathscr{F}, T_m)$ is a Menger space, called an *E-space*. Further, every metrically generated space is isometric to an E-space.

A *topologically generated triple* (TG triple) is an ordered triple $(X, \mathscr{T}, P)$ such that

(i) X is a non-empty set;

(ii) $\mathscr{T} = \{\mathscr{T}_\gamma\}_{\gamma \in \Gamma}$ is a collection of topologies on X;

(iii) P is a probability measure on $\mathscr{T}$ such that $P(\mathscr{T}) = 1$ and such that $\{\mathscr{T}_\gamma : x \in \mathrm{Cl}_\gamma(A)\}$ is P-measurable for every x, A. ($\mathrm{Cl}_\gamma(A)$ denotes $\mathrm{Cl}(A)$ relative to $\mathscr{T}_\gamma$.)

Given a TG triple $(X, \mathscr{T}, P)$, define a mapping $\underset{\sim}{c}$ by

$$c(x, A) = P\{\mathscr{T}_\gamma : x \in \mathrm{Cl}_\gamma(A)\} \qquad \text{for each } x, A.$$

It is easily verified that $\underset{\sim}{c}$ satisfies (L1)-(L3). Thus $\underset{\sim}{c}$ determines a left-continuous θ-closure on X, called a TG *θ-closure*; and if (L5) is satisfied with a t-function T, we call (X, θ, T) a *topologically generated space* (TG space). Thus a collection of topologies on which there is a probability measure generates a PT space for which $c(x, A)$ *is* the probability that x is in $\mathrm{Cl}(A)$.

Stevens has shown for metrically generated spaces that in general T_m is the "best possible" t-norm (in the sense of Thorp [8]). The analogue for TG spaces is given by

THEOREM 1. *If θ is a* TG *θ-closure on X, then (X, θ, T_u) is a* TG *space. Further, for any t-function T stronger than T_u, there exists a* TG *θ-closure on a set for which* (C6) *is not satisfied with T.*

Proof. To exhibit (L5) it is sufficient to show that $c(x, A^1) \geqslant \lambda$ implies $c(x, A) \geqslant T_u(\lambda, 1) = \lambda$. Let

$$B = A^1 = [y : P\{\mathscr{T}_\gamma : y \in \mathrm{Cl}_\gamma(A)\} = 1],$$

and suppose $c(x, B) \geqslant \lambda$. Since

$$P\{\mathscr{T}_\gamma : \mathrm{Cl}_\gamma(B) \subseteq \mathrm{Cl}_\gamma(A)\} = P\{\mathscr{T}_\gamma : B \subseteq \mathrm{Cl}_\gamma(A)\} = 1$$

and $A \subseteq B$,

$$P\{\mathscr{T}_\gamma : \mathrm{Cl}_\gamma(B) = \mathrm{Cl}_\gamma(A)\} = 1.$$

Thus

$$c(x, A) = P\{\mathscr{T}_\gamma : x \in \mathrm{Cl}_\gamma(A)\} = P\{\mathscr{T}_\gamma : x \in \mathrm{Cl}_\gamma(B)\} = c(x, B) \geqslant \lambda.$$

To prove the second assertion, let T be any t-function stronger than T_u, and let (λ_0, μ_0) be a pair for which $T(\lambda_0, \mu_0) > T_u(\lambda_0, \mu_0)$.

Case 1. $\mu_0 = 1$. Then $T(\lambda_0, 1) > \lambda_0$. Let $X = \{1, 2, 3\}$, $\mathscr{T}_1 = [\varnothing, X, \{3\}]$, $\mathscr{T}_2 = [\varnothing, X, \{2, 3\}]$, $P(\mathscr{T}_1) = \lambda_0$, $P(\mathscr{T}_2) = 1 - \lambda_0$. Since

$$(\{1\}^1)^{\lambda_0} = \{1\}^{\lambda_0} \supset \{1\} = \{1\}^\nu$$

for every $\nu > \lambda_0$, (C6) is not satisfied with $T(\lambda_0, 1) = \nu > \lambda_0$.

Case 2. $\mu_0 < 1$. Then $T(1, \mu_0) \geqslant T(\lambda_0, \mu_0) = \nu_0 > 0$. Choose an integer n sufficiently large that $1 - 1/n \geqslant \mu_0$. Let $X = \{0, 1, 2, \ldots, n + 1\}$, $\mathscr{T}_i = [\varnothing, X, \{0, i + 1\}]$, and $P(\mathscr{T}_i) = 1/n$ for $i = 1, 2, \ldots, n$. Since $\mathrm{Cl}_i(\{1\}) = \{1, 2, \ldots, i, i + 2, \ldots, n + 1\}$ for each i,

$$\{1\}^\mu = \begin{cases} X & \mu = 0 \\ \{1, 2, \ldots, n + 1\} & 0 < \mu \leqslant 1 - \dfrac{1}{n}. \\ \{1\} & 1 - \dfrac{1}{n} < \mu \leqslant 1 \end{cases}$$

But since $\mathrm{Cl}_i(\{1, 2, \ldots, n + 1\}) = X$ for each i, $(\{1\}^\mu)^1 = X$ for $\mu \leqslant 1 - 1/n$. Since $0 \notin \{1\}^\nu$ for any $\nu \neq 0$, $(\{1\}^{\mu_0})^1 = X \supset \{1\}^{\nu_0}$, whence (C6) is not satisfied with $T(1, \mu_0) \geqslant \nu_0$.

Note that 3.2 shows that not every PT space can be topologically generated.

We exhibit two constructions of θ-closures each of which is equivalent to that of the TG θ-closure. The verification of (L1)-(L3) is direct in each case. We omit the proofs of 5.2 and 5.3 because they are somewhat lengthy and are similar to the proof of the analogous results for metrically generated spaces in [6].

Let $H = \{h_\gamma\}_{\gamma \in \Gamma}$ be a collection of functions from a set X into a topological space $(Y, \mathscr{S})$ and P be a probability measure on H such that $\{h_\gamma : h_\gamma(x) \in \mathrm{Cl}(h_\gamma(A))\}$ is P-measurable for every x, A. If we define $\underset{\sim}{c}$ by $c(x, A) = P\{h_\gamma : h_\gamma(x) \in \mathrm{Cl}(h_\gamma(A))\}$ for each x, A, then $\underset{\sim}{c}$ determines a left-continuous θ-closure on X, called a *θ_H-closure* on X.

THEOREM 2. *If θ is a θ_H-closure on X, then there exists a* TG *triple $(X, \mathscr{T}, P')$ for which θ is a* TG *θ-closure on X. Conversely, if θ is a* TG *θ-closure on X, then there exists a collection of functions H from X into a topological space $(Y, \mathscr{S})$ for which θ is a θ_H-closure on X.*

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $X = \{x_\gamma\}_{\gamma \in \Gamma}$ be a collection of functions from Ω into a topological space $(E, \mathscr{U})$. For $A \subseteq X$, $t \in \Omega$, let $E(t, A) = \mathrm{Cl}(\{x_\gamma(t) : x_\gamma \in A\})$. If we define $\underline{c}$ by

$$c(x_\gamma, A) = P\{t : x_\gamma(t) \in E(t, A)\}$$

for each γ, A, then $\underline{c}$ determines a left-continuous θ-closure on X, called a *θ_E-closure* on X.

THEOREM 3. *If θ is a θ_E-closure on X, then there exists a* TG *triple $(X, \mathscr{T}, P')$ for which θ is a* TG *θ-closure on X. Conversely, let θ be a* TG *θ-closure on a set X and P' be the probability measure on Γ induced from P by the function $\psi(\mathscr{T}_\gamma) = \gamma$ for each γ. Then there exists a collection of functions $F = \{f_x\}_{x \in X}$ from the probability space induced by ψ into a topological space $(E, \mathscr{U})$ such that for each $A \subseteq X$, λ, $\theta(A, \lambda) = \theta_E'(f_A, \lambda)$, where θ_E' is the θ_E-closure on F.*

Next we present conditions on a TG θ-closure which are sufficient to guarantee (C6) for the t-functions T_w, T_m, and Min.

THEOREM 4. *If θ is a* TG *θ-closure on X, then (X, θ, T_w) is a* TG *space iff for every $A \neq \varnothing$, μ, $\bigcap_{\gamma \in \Gamma'} \mathrm{Cl}_\gamma(A^\mu) = A^\mu$ for every subset Γ' of Γ such that $P\{\mathscr{T}_\gamma : \gamma \in \Gamma'\} = 1$.*

Proof. First, by 5.1 it is sufficient to verify (C6) for $T_w(1, \mu)$. For any μ, $y \in (A^\mu)^1$ iff $P\{\mathscr{T}_\gamma : y \in \mathrm{Cl}_\gamma(A)\} = 1$ iff $y \in \bigcap_{\gamma \in \Gamma'} \mathrm{Cl}_\gamma(A^\mu)$ for some such Γ' iff $y \in A^\mu$. Hence $(A^\mu)^1 = A^\mu$.

Conversely, if $\bigcap_{\gamma \in \Gamma'} \mathrm{Cl}_\gamma(A^\mu) \supset A^\mu$ for some $A \neq \varnothing$ and some such Γ', then $(A^\mu)^1 \supseteq \bigcap_{\gamma \in \Gamma'} \mathrm{Cl}_\gamma(A^\mu) \supset A^\mu$, whence (C6) is not satisfied with $\mu = T_w(1, \mu)$.

THEOREM 5. *If θ is a* TG *θ-closure on X and for every $A \neq \varnothing$, μ*

$$P\{\mathscr{T}_\gamma : \mathrm{Cl}_\gamma(A^\mu) = A^\mu \cup \mathrm{Cl}_\gamma(A)\} \geqslant \mu,$$

then (X, θ, T_m) is a TG *space.*

Proof. (Note that in every case $\mathrm{Cl}_\gamma(A^\mu) \supseteq A^\mu \cup \mathrm{Cl}_\gamma(A)$.) Let $B = A^\mu$, and suppose $c(x, B) \geqslant \lambda$.

Case 1. $x \in B$. Then $c(x, A) \geqslant \mu \geqslant T_m(\lambda, \mu)$.

Case 2. $x \notin B$. If $x \notin \mathrm{Cl}_\gamma(A)$ and $x \in \mathrm{Cl}_\gamma(B)$, then $\mathrm{Cl}_\gamma(B) \supset \mathrm{Cl}_\gamma(A)$, and thus $\mathrm{Cl}_\gamma(B) \supset B \cup \mathrm{Cl}_\gamma(A)$. Hence

$$\{\mathcal{T}_\gamma : x \notin \mathrm{Cl}_\gamma(A)\} \subseteq \{\mathcal{T}_\gamma : \mathrm{Cl}_\gamma(B) \supset B \cup \mathrm{Cl}_\gamma(A)\} \cup \{\mathcal{T}_\gamma : x \notin \mathrm{Cl}_\gamma(B)\}.$$

Thus we have

$$\begin{aligned} 1 - c(x, A) &= P\{\mathcal{T}_\gamma : x \notin \mathrm{Cl}_\gamma(A)\} \\ &\leqslant P\{\mathcal{T}_\gamma : \mathrm{Cl}_\gamma(B) \supset B \cup \mathrm{Cl}_\gamma(A)\} + P\{\mathcal{T}_\gamma : x \notin \mathrm{Cl}_\gamma(B)\} \\ &\leqslant (1 - \mu) + (1 - \lambda), \end{aligned}$$

whence $c(x, A) \geqslant \lambda + \mu - 1$.

COROLLARY 5.1. *If θ is a* TG *θ-closure on X and $\mathcal{T}_\gamma$ is a pseudo-metric topology for every γ, then (X, θ, T_m) is a* TG *space.*

Proof. It is sufficient to show that $P\{\mathcal{T}_\gamma : \mathrm{Cl}_\gamma(A^\mu) \subseteq \mathrm{Cl}_\gamma(A)\} \geqslant \mu$ for every $A \neq \varnothing, \mu$. Since $x \in \mathrm{Cl}_\gamma(A)$ iff $d_\gamma(x, A) = 0$, $P\{\mathcal{T}_\gamma : d_\gamma(y, A) = 0\} \geqslant \mu$ for every $y \in A^\mu$. But $d_\gamma(x, A) \leqslant d_\gamma(x, y) + d_\gamma(y, A)$ for every x, y, A. Then for any $y \in A^\mu$, $P\{\mathcal{T}_\gamma : d_\gamma(x, A) \leqslant d_\gamma(x, y)\} \geqslant P\{\mathcal{T}_\gamma : d_\gamma(y, A) = 0\} \geqslant \mu$, whence $P\{\mathcal{T}_\gamma : d_\gamma(x, A) \leqslant \inf_{y \in A^\mu} d_\gamma(x, y)\} \geqslant \mu$. But if $x \in \mathrm{Cl}_\gamma(A^\mu)$ and $d_\gamma(x, A) \leqslant \inf_{y \in A^\mu} d_\gamma(x, y)$, then $d_\gamma(x, A) = 0$, whence $x \in \mathrm{Cl}_\gamma(A)$. Hence

$$P\{\mathcal{T}_\gamma : \mathrm{Cl}_\gamma(A^\mu) \subseteq \mathrm{Cl}_\gamma(A)\} \geqslant P\{\mathcal{T}_\gamma : d_\gamma(x, A) \leqslant \inf_{y \in A^\mu} d_\gamma(x, y)\} \geqslant \mu.$$

THEOREM 6. *If θ is a* TG *θ-closure on X and for every $A \neq \varnothing, \mu$*

$$P\{\mathcal{T}_\gamma : \mathrm{Cl}_\gamma(A^\mu) = A^\mu \cup \mathrm{Cl}_\gamma(A)\} = 1,$$

then $(X, \theta, \mathrm{Min})$ is a TG *space.*

Proof. Let $B = A^\mu$, and suppose $c(x, B) \geqslant \lambda$.

Case 1. $x \in B$. Then $c(x, A) \geqslant \mu \geqslant \mathrm{Min}(\lambda, \mu)$.

Case 2. $x \notin B$. Then

$$\begin{aligned} c(x, A) &= P\{\mathcal{T}_\gamma : x \in \mathrm{Cl}_\gamma(A)\} \\ &= P\{\mathcal{T}_\gamma : x \in \mathrm{Cl}_\gamma(B)\} = c(x, B) \geqslant \lambda \geqslant \mathrm{Min}(\lambda, \mu). \end{aligned}$$

COROLLARY 6.1. *If θ is a* TG *θ-closure on X and $\mathcal{T}$ is linearly ordered by inclusion, then $(X, \theta, \mathrm{Min})$ is a* TG *space.*

Proof. It is sufficient to show that $\mathrm{Cl}_\gamma(A^\mu) = A^\mu \cup \mathrm{Cl}_\gamma(A)$ for every $A \neq \varnothing, \mu, \gamma$. If $\mathcal{T}_i \geqslant \mathcal{T}_j$, then for any A, $\mathrm{Cl}_i(A) \subseteq \mathrm{Cl}_j(A)$. Thus for any μ

there exist Γ_1, Γ_2 with $\Gamma_1 \cup \Gamma_2 = \Gamma$ such that $\mathrm{Cl}_i(A) \subseteq A^\mu \subseteq \mathrm{Cl}_j(A)$ for every $i \in \Gamma_1$, $j \in \Gamma_2$. Let $B = A^\mu$. Since $A \subseteq B \subseteq \mathrm{Cl}_j(A)$, $\mathrm{Cl}_j(B) = \mathrm{Cl}_j(A)$ for every $j \in \Gamma_2$. Since $\mathrm{Cl}_i(B) \subseteq \mathrm{Cl}_j(B) = \mathrm{Cl}_j(A)$ for every i, j, $\mathrm{Cl}_i(B) = B$ for every $i \in \Gamma_1$. Hence $\mathrm{Cl}_\gamma(B) = B \cup \mathrm{Cl}_\gamma(A)$ for every $\gamma \in \Gamma_1 \cup \Gamma_2 = \Gamma$.

If $(X, \mathscr{F})$ is a metrically generated space, then by Stevens' theorem and 4.1

$$c'(x, A) = \begin{cases} \inf\limits_{\alpha>0} \sup\limits_{y\in A} P\{d_\gamma : d_\gamma(x, y) < \alpha\}, & A \neq \varnothing \\ 0, & A = \varnothing \end{cases}$$

determines a θ-closure θ' such that (X, θ', T_m) is a PT space. If, in addition, $\{d_\gamma : d_\gamma(x, A) = 0\}$ is P-measurable for every x, A, then by 5.5.1 (X, θ, T_m) is a TG space with $c(x, A) = P\{d_\gamma : d_\gamma(x, A) = 0\}$. The relationship between these θ-closures is revealed in

PROPOSITION 7. *If $\underline{c}$ and $\underline{c}'$ are defined as above, then $c(x, A) \geqslant c'(x, A)$ for every x, A.*

Proof. By Sherwood's theorem and 5.3 it is sufficient to verify the inequality for an E-space and the associated θ_E-closure. For $A \subseteq X$, $t \in \Omega$, let $A(t) = \{x(t) : x \in A\}$. For any $y \in A$,

$$P\{t : d(x(t), A(t)) < \alpha\} \geqslant P\{t : d(x(t), y(t)) < \alpha\},$$

whence

$$P\{t : d(x(t), A(t)) < \alpha\} \geqslant \sup_{y\in A} P\{t : d(x(t), y(t)) < \alpha\}.$$

But

$$P\{t : d(x(t), A(t)) = 0\} = \inf_{\alpha>0} P\{t : d(x(t), A(t)) < \alpha\}.$$

Thus

$$\begin{aligned} c(x, A) &= P\{t : d(x(t), A(t)) = 0\} \\ &\geqslant \inf_{\alpha>0} \sup_{y\in A} P\{t : d(x(t), y(t)) < \alpha\} = c'(x, A). \end{aligned}$$

Further, we exhibit an E-space for which $c > c'$ for one pair (x, A). Let $\Omega = \Gamma = I$, and let P be the Lebesque measure on Ω. Let $G = \{g_\gamma\}_{\gamma\in\Gamma}$ be the collection of functions on Ω defined by

$$g_\gamma(t) = \begin{cases} \gamma, & t \neq \gamma \\ 0, & t = \gamma. \end{cases}$$

Define a metric on G by

$$d(g_i(t), g_j(t)) = \begin{cases} 1, & g_i(t) \neq g_j(t) \\ 0, & g_i(t) = g_j(t). \end{cases}$$

For $\gamma \neq 0$,

$$F_{g_0 g_\gamma}(\alpha) = P\{t : d(g_0(t), g_\gamma(t)) < \alpha\} = \begin{cases} P(\gamma) = 0, & \alpha \leqslant 1 \\ P(\Gamma) = 1, & \alpha > 1. \end{cases}$$

Let $A = \{g_\gamma : \gamma \in (0, 1)\}$. Then $c(g_0, A) = 1 > 0 = c'(g_0, A)$. (This example is due to H. Sherwood.)

6. The θ^*-Interior Operator

There are several equivalent ways to define the structure of a PT space. For example, if θ is a θ-closure on X, then the *θ^*-interior* on X determined by θ is the mapping $\theta^* : \mathscr{P}(X) \times I \to \mathscr{P}(X)$ (we write $\theta^*(A, \lambda) = {}^\lambda A$) given by

$${}^\lambda A = X - (X - A)^{1-\lambda}, \qquad \text{for each } A, \lambda.$$

It is easy to verify that ${}^\lambda X = X$ for every $\lambda < 1$, ${}^1 A = \phi$ for every A, ${}^\lambda A \subseteq A$ for every A, λ, ${}^\lambda A \subseteq {}^\lambda B$ for every λ whenever $A \subseteq B$, and $\lambda \leqslant \mu$ implies ${}^\lambda A \supseteq {}^\mu A$ for every A. Conversely, given a mapping with these properties, there exists a unique θ-closure on X such that $A^\lambda = X - {}^{1-\lambda}(X - A)$ for every A, λ. In addition, θ satisfies (C6) with t-function T iff θ^* satisfies ${}^\lambda({}^\mu A) \supseteq {}^{T^*(\lambda,\mu)}A$ for every $A \subset X$ and every pair (λ, μ), where T^* is the t-function given by $T^*(\lambda, \mu) = 1 - T(1 - \lambda, 1 - \mu)$. The corresponding probabilistic interpretation is "x is in ${}^\lambda A$ iff the probability that x is in $\mathrm{Int}(A)$ is greater than λ."

Define the mapping $\underset{\sim}{v}$ by $v(x, A) = \sup\{\lambda : x \in {}^\lambda A\}$ for each x, A. Since it can be shown that $v(x, A) + c(x, X - A) = 1$ for every x, A, there is a characterization of PT spaces by fuzzy sets (called *interior clouds*) constructed in a manner analogous to that of 2.3.1.

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